ME-221
SOLUTIONS FOR PROBLEM SET 6

Problem 1

a) The signal $f_1(t)$ can be written as the sum of the two signals f_{1_1} and f_{1_2}:

$$f_{1_1}(t) = \begin{cases} 0 & t < 0 \\ Asin(\omega t) & t \geq 0 \end{cases}$$

$$f_{1_2}(t) = \begin{cases} 0 & t < \pi/\omega \\ -Asin(\omega t) = Asin(\omega t - \pi) & t \geq \pi/\omega \end{cases}$$

These two signals correspond to a translation of $\tau = \pi/\omega$ as shown in Figure 1:

$f_{1_2} = f_{1_1}(t - \tau)$

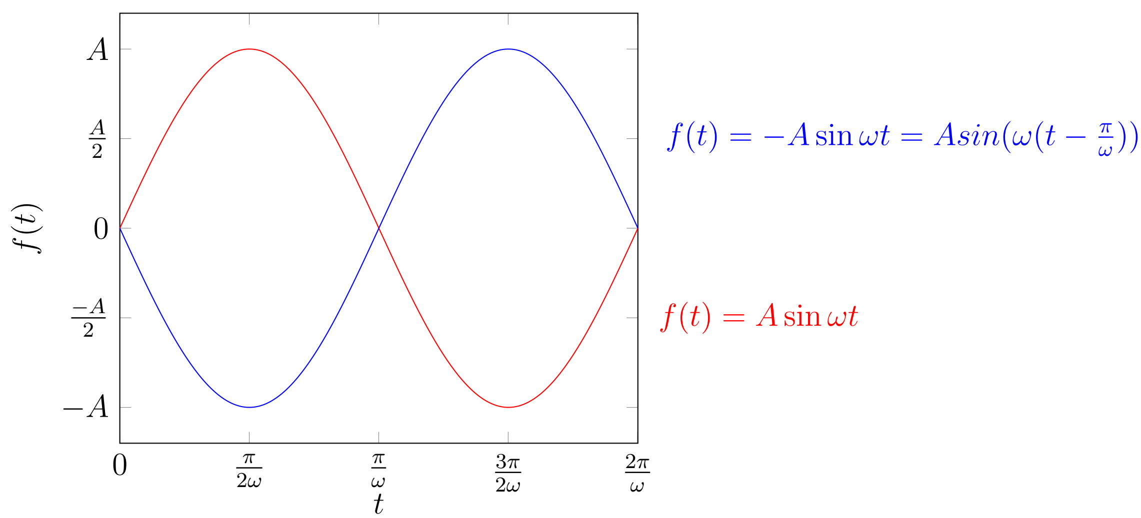


Figure 1: Signals with translation

Using the linearity and time-shift property of the Laplace transform, we can calculate F_1(s):

$$\begin{aligned} F_1(s) &= L[f_1(t)] = L[f_{1_1}(t) + f_{1_2}(t)] \\ &= L[f_{1_1}(t) + f_{1_1}(t - \tau)] \\ &= L[f_{1_1}(t)] + exp(-s\tau)L[f_{1_1}(t)] \end{aligned}$$

Finally:

$$F_1(s) = \frac{A\omega}{s^2+\omega^2}[1 + exp(-s\tau)]$$

b) Using the definition of the Laplace transform:

$$\begin{aligned} F_2(s) &= \int_0^\infty f_2(t)e^{-st}dt = \int_0^T (1+e^{\alpha t})e^{-st}dt - \int_T^\infty e^{(\alpha-s)t}dt \\ &= \frac{1}{s}(1-e^{-sT}) + \frac{1}{\alpha - s}(2e^{(\alpha-s)T} - 1) \end{aligned}$$

for $Re(s) > (\alpha)$

c) Remember that the Laplace transform of a signal $f(t)$ multiplied by t is given by

$$L[t \cdot f(t)] = -\frac{dF(s)}{ds}$$

Using this relation:

$$F_3(s) = L[tsin(\omega t)] = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

Problem 2

a) From Newton's movement law:

$$\begin{aligned} m\ddot{x}_1 &= F - kx_1 - k(x_1 - x_2) \\ m\ddot{x}_2 &= k(x_1 - x_2) - f\dot{x}_2 \end{aligned}$$

Applying now the Laplace transform we obtain:

$$\begin{aligned} ms^2X_1(s) &= F(s) - kX_1(s) - kX_1(s) + kX_2(s) \\ ms^2X_2(s) &= kX_1(s) - kX_2(s) - fsX_2(s) \end{aligned}$$

To find the transfer function $G(s)$ we combine both equations:

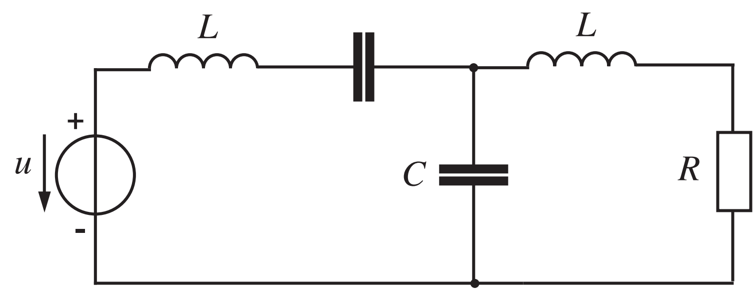
$$\begin{aligned} (ms^2 + fs + k)X_2(s) &= kX_1(s) \\ (ms^2 + 2k)X_1(s) - kX_2(s) &= F(s) \end{aligned}$$

$$\rightarrow G(s) = \frac{X_2(s)}{F(s)} = \frac{k}{m^2s^4 + mfs^3 + 3mks^2 + 2kfs + k^2}$$

b) We can use the following correspondence between paramaters of electrical and mechanical systems:

$$m \leftrightarrow L$$
$$\frac{1}{k} \leftrightarrow C$$
$$f \leftrightarrow R$$

The analogous electrical circuit is:



c) The order of the system is 4, thus we have to define 4 state variables.

Problem 3

a) Because we already have the transforms of $sin\omega t$ and $cos\omega t$, we can use the following trigonometric identity to obtain our answer:

$$sin(\omega t + \phi) = sin(\omega t)cos(\phi) + cos(\omega t)sin(\phi)$$

Laplace transform is a linear operation. Therefore:

$$\begin{aligned}
\mathcal{L}[Asin(\omega t + \phi)] &= A\int_0^\infty sin(\omega t)cos(\phi)e^{-st}dt + A\int_0^\infty sin(\phi)cos(\omega t)e^{-st}dt \\
&= Acos\phi\mathcal{L}(sin\omega t) + Asin\phi\mathcal{L}(cos\omega t) \\
&= Acos\phi\frac{\omega}{s^2+\omega^2} + Asin\frac{s}{s^2+\omega^2}
\end{aligned}$$

Combining these terms gives the answer:

$$= \mathcal{L}[Asin(\omega t + \phi)] = A\frac{s\ sin\phi + \omega\ cos\phi}{s^2+\omega^2}$$

b) Comparing this with $F(s)$ we see that $\omega = 2\sqrt{3}$ and $A(s\ sin\phi + \omega\ cos\phi) = s + 6$. Therefore, $Asin\phi = 1$ and $A\omega\ cos\phi = 2\sqrt{3}Acos\phi = 6$, or:

$$1 = Asin\phi$$
$$6 = 2\sqrt{3}Acos\phi$$

Because $A > 0$, these equations reveal that $sin\phi > 0$ and $cos\phi > 0$. Therefore, ϕ is in the first quadrant ($0 \leq \phi \leq \pi/2\ rad$) and can be computed as follows:

$$\phi = tan^{-1}\frac{sin\phi}{cos\phi} = tan^{-1}\frac{1/A}{6/2\sqrt{3}A} = tan^{-1}\frac{1}{\sqrt{3}} = 0.5236\ rad$$

To find A, we use the fact that $sin^2\phi + cos^2\phi = 1$ for *any* angle ϕ:

$$sin^2\phi + cos^2\phi = (\frac{1}{A})^2 + (\frac{\sqrt{3}}{A})^2 = 1$$

It follows that $A = 2$ and $f(t) = 2sin(2\sqrt{3}t + 0.5236)$

c) Following the same procedure and using the fact that:

$$\mathcal{L}[e^{-at}sin\omega t] = \frac{\omega}{(s+a)^2 + \omega^2}$$

We see that:

$$\begin{aligned}\mathcal{L}[Ae^{-at}sin(\omega t + \phi)] &= Acos\phi\mathcal{L}[e^{-at}sin\omega t] + Asin\phi\mathcal{L}[e^{-at}cos\omega t] \\ &= Acos\phi\frac{\omega}{(s+a)^2 + \omega^2} + Asin\phi\frac{s+a}{(s+a)^2 + \omega^2}\end{aligned}$$

Combining these terms gives the answer:

$$\mathcal{L}[Ae^{-at}sin(\omega t + \phi)] = A\frac{s\ sin\phi + a\ sin\phi + \omega\ cos\phi}{(s+a)^2 + \omega^2}$$

Problem 4

a) Applying the Laplace transformation we obtain:

$$\begin{aligned}&\dot{y}(t) \rightarrow sY(s) - y(0) = sY(s) - 2 \\ &\ddot{y}(t) \rightarrow s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s) - 2s - 1 \\ &\dddot{y}(t) \rightarrow s^3Y(s) - s^2y(0) - s\dot{y}(0) - \ddot{y}(0) = s^3Y(s) - 2s^2 - s - 0.5\end{aligned}$$

Therefore:

$$Y(s) = \frac{2s^2 + 17s + 42.5}{s^3 + 8s^2 + 17s + 10}$$

b) Applying the initial value theorem, we obtain:

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{2s^3 + 17s^2 + 42.5s}{s^3 + 8s^2 + 17s + 10} = 2$$

If $z(t) = \dot{y}(t)$, then:

$$\begin{aligned}
\mathcal{L}[z(t)] &= Z(s) = sY(s) - y(0) \\
\lim_{t\to 0} z(t) &= \lim_{s\to\infty} sZ(s) = \lim_{s\to\infty} s[sY(S) - y(0)] \\
&= \lim_{s\to\infty} s[\frac{2s^3 + 17s^2 + 42.5s}{s^3 + 8s^2 + 17s + 10} - 2] \\
&= \lim_{s\to\infty} s[\frac{s^2 + 8.5s - 20}{s^3 + 8s^2 + 17s + 10}] = 1
\end{aligned}$$

Problem 5

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2 + s + 2} \leftrightarrow s^2Y(s) + sY(s) + 2Y(s) = sU(s) + U(s)$$

Then Laplace,

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = \frac{du}{dt} + u$$

We have an input derivative (see Lecture 4, slide 17).

$$\begin{aligned}
&a_1 = 1, \quad a_2 = 2, \quad b_0 = 0, \quad b_1 = 1, \quad and \quad b_2 = 1 \\
&\dot{x}_2 = -2x_1 - x_2 \\
&\dot{x}_1 = x_2 + u \\
&y = x_1
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
D &= 0
\end{aligned}$$